Review

Euler: $y_{n+1} = y_n + f(t_n, y_n)h$

Improved Euler: $y_{n+1} = y_n + \frac{1}{2}\left[f(t_n, y_n) + f(t_{n+1}, y_{n+1})\right]h$

$y_{n+1} = y_n + f(t_n, y_n)h$
for right side

no RK4 on exam

resonance: input freq = natural freq.

$mx'' + kx = f(t) = \sin(\omega_0 t)$

natural freq. $\sqrt{\frac{k}{m}}$

input freq. ω_0

if $\omega_0 = \sqrt{\frac{k}{m}}$
resonance
(amplitude grows w/o bound)

cosine/sine series

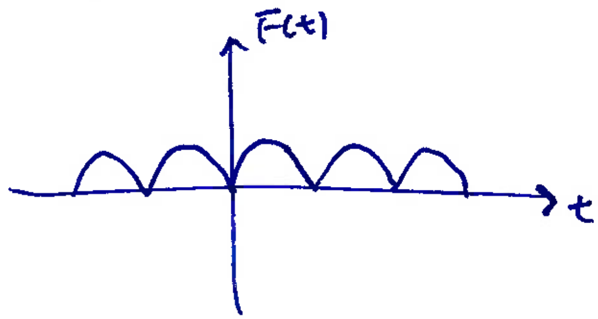
cosine : no b_n (no sine) function is even

Sine : no a_n (no cosine) function is odd

usually we have to add extensions

(# lie of review) for example, $F(t) = \sin(3t)$ $0 < t < \pi$

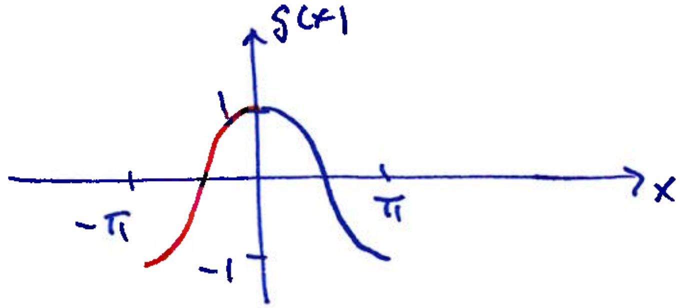
cosine series representation



if expanded as a cosine series, if $a_3 \neq 0$ then there is still a $\cos(3t)$ term, which still causes resonance

$$a_3 = \frac{2}{\pi}\int_0^{\pi} \sin(3t)\cos(3t)\,dt = \cdots = 0$$

\# 9 (review) $g(x) = \cos(x)$ $0 < t < \pi$



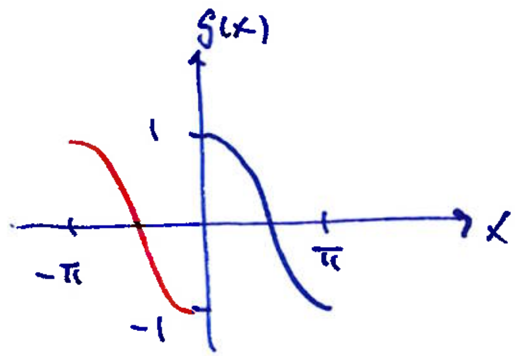
want cosine series, add even extension

normal cos(x) function

a) Fourier cosine series $\frac{1}{2}a_0 + a_1\cos(x) + a_2\cos(2x) + a_3\cos(3x) + \cdots$

$$a_0 = 0,\ a_1 = 1,\ a_2 = a_3 = a_4 = \cdots = 0$$

b)



if we want sine series

$a_n = 0$ for all n

$$b_n = \frac{2}{\pi}\int_0^{\pi} \cos(x)\sin(nx)\,dx$$

$$\sin(A)\cos(B) = \frac{1}{2}\left[\sin(A+B) + \sin(A-B)\right]$$

$$b_n = \frac{2}{\pi}\int_0^{\pi} \frac{1}{2}\left[\sin((n+1)x) + \sin((n-1)x)\right]dx$$

$$= \ldots$$

Fourier series converges to $f(t)$ wherever it is continuous

Fourier series converges to $\dfrac{f(t^-)+f(t^+)}{2}$ if discontinuous at t

Boundary-Value Problem

$x'' + 5x = F(t)$ $\quad$ $x(0) = x(\pi) = 0$ $\quad$ fix end positions

or $x'(0) = x'(\pi) = 0$ $\quad$ fix end velocities

$F(t) = 3 \quad 0 < t < \pi$

add the correct extensions to $F(t)$ based on the BC's

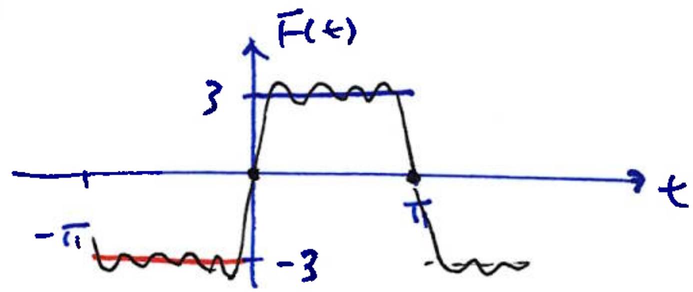
if $x(0)=x(\pi)=0 \rightarrow$ odd extensions $\rightarrow$ sine series

if $x'(0)=x'(\pi)=0 \rightarrow$ even extensions $\rightarrow$ cosine series

if $x''+5x=F(t)$ $\quad x(0)=x(\pi)=0$

$F(t)=3 \quad 0<t<\pi$

add odd extensions



sine series meets the BC's

expand $F(t)=3$ as sine series w/ $L=\pi$

$a_n=0$ for all n

$$b_n=\frac{2}{\pi}\int_0^{\pi} 3\sin(nt)\,dt = \frac{6\left[1-(-1)^n\right]}{n\pi}$$

$$F(t) = \sum_{n=1}^{\infty} \frac{6[1-(-1)^n]}{n\pi} \sin(nt)$$

now assume the particular solution $x(t)$ is the same kind series w/ unknown coefficients

$$x(t) = \sum_{n=1}^{\infty} B_n \sin(nt)$$

sub into $x'' + 5x = F(t)$

$$= \sum_{n=1}^{\infty} \frac{6[1-(-1)^n]}{n\pi} \sin(nt)$$

$$x'(t) = \sum_{n=1}^{\infty} n B_n \cos(nt)$$

$$x''(t) = \sum_{n=1}^{\infty} -n^2 B_n \overset{\sin}{\cancel{\cos}}(nt)$$

$$\sum_{n=1}^{\infty} -n^2 B_n \overset{\sin}{\cancel{\cos}}(nt) + \sum_{n=1}^{\infty} 5 B_n \sin(nt) = \sum_{n=1}^{\infty} \frac{6[1-(-1)^n]}{n\pi} \sin(nt)$$

for each n, $-n^2 B_n + 5B_n = \dfrac{6[1-(-1)^n]}{n\pi}$

$$B_n = \frac{6[1-(-1)^n]}{n\pi(5-n^2)}$$

Fourier Series and Differential Equations Reference

Term	Formula	Period
$f(x)$	$\frac{a_0}{2}+\sum_{n=1}^{\infty}\left(a_n\cos\left(\frac{n\pi x}{L}\right)+b_n\sin\left(\frac{n\pi x}{L}\right)\right)$	$2L$
a_n	$\frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx$	$2L$
b_n	$\frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx$	$2L$
Even Extension	$b_n=0, \quad a_n=\frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx$	$2L$
Odd Extension	$a_n=0, \quad b_n=\frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx$	$2L$

Trigonometric Identities

- $\sin(A)\cos(B)=\frac{1}{2}[\sin(A+B)+\sin(A-B)]$
- $\cos(A)\cos(B)=\frac{1}{2}[\cos(A+B)+\cos(A-B)]$
- $\cos(n\pi)=(-1)^n, \quad \sin(n\pi)=0$

Integration

- $\int u\,dv = uv-\int v\,du$

Common Trigonometric Integrals (for integer m,n)

- $\displaystyle\int_0^{\pi}\sin(nx)\cos(mx)\,dx=\begin{cases}\frac{2n}{n^2-m^2} & \text{if } n-m \text{ is odd}\\ 0 & \text{if } n-m \text{ is even}\end{cases}$
- $\displaystyle\int_{-L}^{L}\cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi x}{L}\right)dx=\begin{cases}L & \text{if } n=m\neq 0\\ 0 & \text{if } n\neq m\end{cases}$
- $\displaystyle\int_{-L}^{L}\sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)dx=\begin{cases}L & \text{if } n=m\neq 0\\ 0 & \text{if } n\neq m\end{cases}$
- $\displaystyle\int_{-L}^{L}\sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi x}{L}\right)dx=0 \quad \text{for all } n,m$

Standard Form Integrals

- $\displaystyle\int x\cos(ax)\,dx=\frac{\cos(ax)}{a^2}+\frac{x\sin(ax)}{a}+C$
- $\displaystyle\int x\sin(ax)\,dx=\frac{\sin(ax)}{a^2}-\frac{x\cos(ax)}{a}+C$